An anisotropic external Serrin problem

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The original Serrin Problem

Let Ω be a smooth domain in $\mathbb{R}^N$ and u a regular solution to the torsion problem

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$|\nabla u_\Omega| = C \qquad \text{on } \partial\Omega.$$

Then Ω must be a **ball** and u_Ω is radial.

[J. Serrin, Arch. Rat. Mech. Anal. 1971]

Classical external Serrin type problem

Let Ω be a smooth domain in $\mathbb{R}^N$ and u a regular solution to the exterior problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u \to 0 & \text{if } |x| \to +\infty, \\ |Du| = C & \text{on } \partial\Omega. \end{cases}$$

Then Ω must be a ball and u_Ω is radial.

[W. Reichel, Arch. Rational Mech. Anal. 1997]

Anisotropic external Serrin type problem

Theorem. Let $H : \mathbb{R}^N \to \mathbb{R}$ be a smooth **norm** such that the unitary ball B_H is strictly convex. Let Ω be a smooth, strictly convex domain in $\mathbb{R}^N$ and u a regular solution to the exterior problem

$$\begin{cases} \Delta_H u = 0, & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ u = 1, & \text{on } \partial\Omega, \\ u(x) \to 0, & \text{for } |x| \to +\infty, \\ H(Du) = C, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

Then Ω must be a H_0-**ball** and u_Ω is radial w.r.t. the dual norm H_0.

[CB, G. Ciraolo, P. Salani]

$\rightsquigarrow \Delta_H u$ is the **Finsler Laplacian** operator: $\Delta_H u = div(H(Du)\nabla_\xi H(Du))$.

Ingredients: norms

Let $H : \mathbb{R}^N \to \mathbb{R}$ be the support function of a convex body K
$\rightsquigarrow H$ defines a norm in $\mathbb{R}^N$:

- **(i)** H is convex;
- **(ii)** $H(\xi) \geq 0$ for $\xi \in \mathbb{R}^N$ and $H(\xi) = 0$ if and only if $\xi = 0$;
- **(iii)** $H(t\xi) = |t|H(\xi)$ for $\xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$. $\rightsquigarrow H$ is 1-homog.

Def. The **dual norm:** $H_0(x) = \sup_{\xi \neq 0} \dfrac{x \cdot \xi}{H(\xi)}, \quad x \in \mathbb{R}^N.$

$\rightsquigarrow H_0$ is the support function of K^* (the polar body of K).

Notice: $x \in \Omega \subseteq \mathbb{R}^N$; $Du(x) \in$ dual space of $\mathbb{R}^N$
H_0 norm of $\mathbb{R}^N$; H norm of $\mathbb{R}^N$ seen as dual space.

Ingredients: Finsler Capacity (i)

Def. The Finsler capacity of a convex, bounded, open set $\Omega \subset \mathbb{R}^N$ is

$$Cap_H(\Omega) = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^n} H^2(Dv)dx : \ v \in C_c^\infty(\mathbb{R}^n), \ v_{|\Omega} \geq 1 \right\}.$$

Def. We call u the Finsler capacitary potential of Ω the solution to

$$\begin{cases} \Delta_H u = 0, & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u = 1, & \text{on } \partial\Omega, \\ u(x) \to 0, & \text{for } H_0(x) \to +\infty, \end{cases}$$

$\rightsquigarrow$ if $\Omega = rB_{H_0}$ (H_0 ball of radius r) then

$$u(x) = \frac{H_0^{2-n}(x)}{r^{2-n}},$$

$x \in \mathbb{R}^N \setminus rB_{H_0}$.

Ingredients: Finsler Capacity (ii)

Weak comparison principle: let E be a bounded domain and assume that

$$-\Delta_H u \leq -\Delta_H v \text{ in } E, \text{ and } u \leq v \text{ on } \partial E, \text{ then}$$
$$u \leq v \quad \text{a.e. in } E.$$

Maximum principle: if $\Delta_H u = 0$ in E, then

$$\min_{\partial E} u \leq u(x) \leq \max_{\partial E} u \text{ a.e. in } E.$$

[V. Ferone, B. Kawohl, Proc. AMS 2009]

Ingredients: anisotropic geometric space

Def. Anisotropic Perimeter $P_H(\Omega) = \int_\Omega H(\nu)\, d\mathcal{H}^{N-1}(x)$.

Def. Ω has Wulff Shape if $\Omega = \{H_0(x) \le C\}$.

$\rightsquigarrow$**Wulff Thm:** $P_H^N(\Omega) \ge C|\Omega|^{N-1}$; $=$ holds iff Ω has Wulff shape.

"Def." The anisotropic mean curvature $\mathrm{H}_H(x)$ extends the classical notion of mean curvature.

$\rightsquigarrow$If $\mathrm{H}_H(\Omega)$ is constant then Ω has Wulff shape.

[He, Li, Ma, Ge, Indiana Univ, 2009]

$\rightsquigarrow \Delta_H v = \mathrm{H}_H \frac{\partial v}{\partial \nu_H} + \frac{\partial^2 v}{\partial \nu_H^2} = \mathrm{H}_H H + H_{\xi_i} H_{\xi_j} v_{ij}$.

[G. Wang, C. Xia. Arch. Ration. Mech. Anal. 2011]

Ingredients: Newton's Inequality

Def. Let A be a symmetric matrix with $\lambda_1, ..., \lambda_N$ eigenvalues.

$$S_2(A) = \sum_{1 \le i_1 \le i_2 \le N} \lambda_{i_1} \lambda_{i_2}.$$

Newton's Inequality.[special case] Let v, V regular functions. Set $W = \nabla^2_\xi V(Dv(x))\, D^2 v(x)$. Then:

$$S_2(W) \le \frac{N-1}{2N}\, tr(W)^2 \tag{0.2}$$

Moreover, if $tr(W) \neq 0$ and equality holds in (0.2), then $W(x) = \gamma(x)\, I$, and $\nabla^2 V$ is, in fact, positive definite.

Main result

Theorem. Let H be a regular strictly convex norm of $\mathbb{R}^N$; let u be a solution to (0.1) in $\mathbb{R}^N \setminus \Omega$. Then Ω has Wulff shape.

Idea of the proof:

Consider $v(x) = u^{\frac{2}{2-n}}$; it solves

$$\begin{cases} \Delta_H v = \dfrac{N}{v} V(Dv) & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ v = 1 & \text{on } \partial\Omega \\ u \to +\infty & \text{if } |x| \to \infty \\ H(Du) = \frac{2}{N-2} C & \text{on } \partial\Omega, \end{cases}$$

where $V(\xi) = \frac{1}{2} H^2(\xi)$.
Denote $V_0(\eta) = \frac{1}{2} H_0^2(\eta)$; $W = \nabla_\xi^2 V(Dv)\, D^2 v$.

aim: prove that $\mathrm{H}_H(\Omega)$ is constant $\rightsquigarrow \Omega$ has Wulff shape.

Idea of the proof (i)

- **Step I:** $C = \dfrac{N-2}{N} \dfrac{P_H(\Omega)}{|\Omega|}$;
- **Step II:** via Newton's Inequality for W it holds:

$$P_H(\Omega) \leq \frac{N-2}{C} \int_{\partial\Omega} \frac{\mathrm{H}_H(\Omega)}{N-1} H(\nu)\, d\mathcal{H}^{N-1}(x).$$

$$\rightsquigarrow P_H^2(\Omega) \leq N|\Omega| \int_{\partial\Omega} \frac{\mathrm{H}_H(x)}{N-1} H(\nu(x))\, d\mathcal{H}^{N-1}(x).$$

- **Step III: Minkowski inequality** Let H be a C^2 norm of $\mathbb{R}^N$ and Ω a regular strictly convex domain of $\mathbb{R}^N$, it holds

$$P_H^2(\Omega) \geq N|\Omega| \int_{\partial\Omega} \frac{\mathrm{H}_H(x)}{N-1} H(\nu(x))\, d\mathcal{H}^{N-1}(x).$$

- **Step IV:** $\rightsquigarrow$ equality holds in Newton's inequality for $W = \nabla_\xi^2 V(Dv(x))\, D^2 v(x)$ that is $W = \gamma(x) I$.

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Idea of the proof (ii)

- **Step V:** $W = \gamma I \rightsquigarrow$ for every $x \in \mathbb{R}^N \setminus \overline{\Omega}$ it holds

$$N\gamma(x) = tr(W) = \Delta_H v = \mathrm{H}_H H + H_{\xi_i} H_{\xi_j} v_{ij};$$

- **Step VI:** $W = \nabla_\xi^2 V\, D^2 v \rightsquigarrow D^2 v = \gamma \nabla_\eta^2 V_0(\nabla H(Dv)) \rightsquigarrow$

$$H_{\xi_i} H_{\xi_j} v_{ij} = \gamma H_{\xi_i} H_{\xi_j} V^0_{\eta_i \eta_j}(\nabla H(Dv)) = 2\gamma V_0(\nabla H) = \gamma;$$

since V_0 is 1-homog and $H_0(\nabla_\xi H) = 1$.

- $\rightsquigarrow$ for every $x \in \mathbb{R}^N \setminus \overline{\Omega}$: $\mathrm{H}_H(x) H(Dv) + \gamma(x) = N\gamma(x)$.
- **Step VII:** let $x \in \partial\Omega \rightsquigarrow H(Dv)$=constant;

$N\gamma = \Delta_H v = NV(Dv)$=constant

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