Stabilité et valeurs propres Wentzell

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ANR Optiform
09/01/2015, Rennes

Outline

1. Stabilité

2. Wentzell eigenvalues
   - Extremal cases $\beta = 0$ and $\beta = \infty$
   - Generalization of Brock's bound
   - First order analysis
   - Second order analysis

Objectif

Partant d'une inégalité isopérimétrique :

$$F(\Omega) \geq F(\Omega^*), \quad \forall \Omega \in \mathcal{A}$$

on cherche à montrer

$$F(\Omega) - F(\Omega^*) \geq \varphi(d(\Omega, \Omega^*)), \quad \forall \Omega \in \mathcal{A}$$

où φ est un module de continuité et d est une distance entre Ω et Ω^*.

Inégalité Isopérimétrique

Exemple : Si R est tel que $|\Omega| = |B_R|$, alors

$$P(\Omega) - P(B_R) \geq c_{d,R} \inf_x \{|\Omega \Delta B_R(x)|\}^2$$

3 méthodes :

1. Par symétrisation (Fusco-Maggi-Pratelli, 2008)
2. Par transport optimal (Figalli-Maggi-Pratelli, 2010)
3. Par utilisation de dérivées secondes (Cicalese-Leonardi, 2012)

Sur la 3ème méthode

Stratégie :

1. Montrer que B est un point critique stable (Fuglede, 1989)
2. Montrer que B est un minimum local strict parmi les voisins réguliers.
3. Montrer que B est un minimum local strict parmi les voisins L^1 (White, Morgan-Ros, Cicalese-Leonardi, Fusco et al.)

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Application : Stabilité pour l'inégalité de Faber-Krahn (Brasco-De Philippis-Velichkov 2013)

Quelques dérivées secondes

Remarque (Novruzi-Pierre 2000) : il existe $\ell_2 = \ell_2[J](\Omega)$ forme bilinéaire symétrique sur $C^\infty(\partial\Omega)$ tel que si $J'(\Omega) = 0$ alors

$$J''(\Omega)(\mathbf{V}, \mathbf{V}) = \ell_2(V_n, V_n), \qquad \text{où} \qquad V_n = \mathbf{V} \cdot n_{|\partial\Omega}$$

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$$\varphi(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} \alpha_{k,l}(\varphi) Y^{k,l}(x), \quad \text{for } |x| = 1.$$

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$$\begin{aligned} \ell_2[\mathrm{Vol}](B_1).(\varphi,\varphi) &= \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} (d-1)\, \alpha_{k,l}(\varphi)^2, \\ \ell_2[E](B_1).(\varphi,\varphi) &= \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \left[\frac{2k-(d+1)}{2d^2}\right] \alpha_{k,l}(\varphi)^2, \end{aligned}$$

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$$\ell_2[P](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} \left[k^2+(d-2)k+(d-1)(d-2)\right]\ \alpha_{k,l}(\varphi)^2,$$

Un exemple (Dambrine-Lamboley 2014)

$$F(\Omega) = P(\Omega) - \gamma E_1(\Omega), \qquad \mathcal{A} = \{\Omega, |\Omega| = V_0\}$$

($\gamma > 0$ petit)

$$E_1(\Omega) = \frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 - \int_\Omega u_\Omega = \inf_{v \in H_0^1(\Omega)} \left\{ \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v \right\}.$$

1. B est un point critique stable,
2. B est un minimum local strict parmi les voisins réguliers
3. B n'est pas un minimum local strict parmi les voisins L^1

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Wentzell eigenvalue problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ -\beta\Delta_\tau u + \partial_n u = \lambda u & \text{on } \partial\Omega \end{cases}$$

where $\beta \in \mathbb{R}_+$.

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Remark : it is the spectrum of

$$-\beta\Delta_\tau + \mathcal{D}$$

where $\mathcal{D}$ is the Dirichlet-to-Neumann operator on $\partial\Omega$.

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Question : *Does the ball maximize $\lambda_{1,\beta}$ among smooth open sets of given volume ?*

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Steklov eigenvalue problem, $\beta = 0$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_n u = \lambda^{St} u & \text{on } \partial\Omega \end{cases}$$

It has a discrete spectrum consisting in a sequence

$$\lambda_0^{St}(\Omega) = 0 < \lambda_1^{St}(\Omega) \leq \lambda_2^{St}(\Omega) \ldots \to +\infty$$

Bound for λ_1^{St}, $\beta = 0$, Dimension d

- Brock (2001) : Ω any smooth set in $\mathbb{R}^d$ such that $\int_{\partial\Omega} x = 0$:

$$\sum_{i=1}^{d} \frac{1}{\lambda_i^{St}(\Omega)} \geq \frac{1}{|\Omega|} \int_{\partial\Omega} |x|^2 \geq d \left(\frac{|\Omega|}{\omega_d}\right)^{\frac{1}{d}}.$$

$$\lambda_1^{St}(\Omega) \leq \frac{d|\Omega|}{\int_{\partial\Omega} |x|^2} \leq \left(\frac{\omega_d}{|\Omega|}\right)^{\frac{1}{d}}.$$

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- Contrainte de périmètre ? (OK si $d = 2$, Weinstock 1954)

Laplace-Beltrami eigenvalue problem, $\beta = +\infty$

$$-\Delta_\tau u = \lambda u \quad \text{on} \quad \partial\Omega$$

$$\lambda_0^{LB}(\partial\Omega) = 0 < \lambda_1^{LB}(\partial\Omega) \leq \lambda_2^{LB}(\partial\Omega) \ldots \to +\infty$$

Remarks :

- $\lambda_1^{LB}(\alpha\partial\Omega) = \alpha^{-2}\lambda_1^{LB}(\partial\Omega)$.
- $\lambda_1^{LB}(\Omega) = \lim_{\beta\to\infty} \frac{\lambda_{1,\beta}(\Omega)}{\beta}$.

Bound for λ_1^{LB}, $\beta = +\infty$, 2-dimensional surface

- Hersch (1970) : If $\Omega \subset \mathbb{R}^3$ smooth and bounded is such that $\partial\Omega$ is diffeomorphic to the 2-dimensional sphere $\mathbb{S}^2 = \partial B$, then

$$\frac{1}{\lambda_1^{LB}(\partial\Omega)} + \frac{1}{\lambda_2^{LB}(\partial\Omega)} + \frac{1}{\lambda_3^{LB}(\partial\Omega)} \geq \frac{3}{8\pi} P(\Omega).$$

$$\lambda_1^{LB}(\partial\Omega) P(\Omega) \leq \lambda_1^{LB}(\mathbb{S}^2) P(B).$$

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- Colbois-Dryden-El Soufi (2009)

$$\sup_{\Omega \subset \mathbb{R}^3} \lambda_1^{LB}(\partial\Omega) P(\Omega) = \infty$$

  where the supremum is taken among smooth compact set Ω.

Bound for λ_1^{LB}, $\beta = +\infty$, *m*-dimensional manifold

- Colbois-Dodziuk (1994) ; for $m \geq 3$,

$$\sup_M \lambda_1^{LB}(M) Vol(M)^{2/m} = \infty$$

where the supremum is taken among smooth compact manifold of dimension m and fixed smooth structure.

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- Colbois-Dryden-El Soufi (2009) ; for $m \geq 3$,

$$\sup_{\Omega \subset \mathbb{R}^{m+1}} \lambda_1^{LB}(\partial\Omega) P(\Omega)^{2/m} = \infty$$

where the supremum is taken among smooth compact set Ω such that $\partial\Omega$ has a fixed smooth structure.

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Generalization of Brock's result

Theorem (Dambrine-Kateb-L. 2014)

Let Ω *a smooth set such that* $\int_{\partial\Omega} x = 0$*. Let* $\Lambda[\Omega]$ *be the spectral radius of* $P[\Omega] = (p_{ij})_{i,j=1,\dots,d}$ *defined as*

$$p_{ij} = \int_{\partial\Omega} (\delta_{ij} - n_i n_j),$$

where $\mathbf{n}$ *is the outward normal vector to* $\partial\Omega$*. Then if* $\beta \geq 0$*, one has :*

$$\sum_{i=1}^{d} \frac{1}{\lambda_{i,\beta}(\Omega)} \geq \frac{\int_{\partial\Omega} |x|^2}{|\Omega| + \beta\Lambda[\Omega]} \tag{1}$$

Equality holds in (1) *if* Ω *is a ball.*

Generalization of Brock's result

Corollary

If $\beta \geq 0$, it holds :

$$\lambda_{1,\beta}(\Omega) \leq d\,\frac{|\Omega| + \beta\Lambda[\Omega]}{\int_{\partial\Omega}|x|^2}$$

Equality holds if Ω is a ball.

- $\beta = 0$ is Brock's result.
- In general, the right-hand side is not maximized by the ball.
- $\beta = \infty$ gives :

$$\lambda_1^{LB}(\partial\Omega) \leq d\,\frac{\Lambda[\Omega]}{\int_{\partial\Omega}|x|^2}$$

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Derivative at the ball

Theorem (Dambrine-Kateb-L. 2014)

Let λ_1 *be the first eigenvalue of the unit ball* B *(order* d*) and* $B_t = (I + t\mathbf{V})(B)$.

Derivative at the ball

Theorem (Dambrine-Kateb-L. 2014)

Let λ_1 be the first eigenvalue of the unit ball B (order d) and $B_t = (I + t\mathbf{V})(B)$. Then there exist d functions $(t \mapsto \lambda_{k,\beta}(t))_{k\in[\![1,d]\!]}$ such that

- $\lambda_{k,\beta}(0) = \lambda_1$,
- *for $|t|$ small, $\lambda_{k,\beta}(t)$ is an eigenvalue of B_t,*
- *the functions $(t \mapsto \lambda_{k,\beta}(t))_{k\in[\![1,d]\!]}$ admit derivatives whose values at 0 are the eigenvalues of the matrix*

$$M_{ij} = C(d,\beta)\int_{\mathbb{S}^{d-1}} V_n x_i x_j .$$

(if $\mathbf{V}$ preserves the volume).

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Sign of the second order derivative

Theorem (Dambrine-Kateb-L. 2014)

Let B be a ball in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $t \mapsto B_t = I + t\mathbf{V} + O(t^2)$ a volume preserving deformation such that

$$V_n \text{ is orthogonal to spherical harmonics of order 2.}$$

Then the functions $(t \mapsto \lambda_{k,\beta}(t))_{k\in[\![1,d]\!]}$ admit a second derivative and there exists $\mu > 0$ such that

$$\sum_{k=1}^{d} \lambda''_{k,\beta}(0) \leq -\mu \|V_n\|^2_{\mathrm{H}^1(\partial B)}.$$

The second order derivative for $B \subset \mathbb{R}^3$

$$\sum_{k=1}^{d} \lambda''_{k,\beta}(0) = -\sum_{k\geq 3} \sum_{l=-k}^{k} F(\beta,k)\ \alpha_{k,l}^2.$$

where $F(\beta, k)$ is the fraction

$$F(\beta,k) = \mu \frac{(k-1)\displaystyle\sum_{m=0}^{3} P_m(k)\beta^m}{k\ (1+\beta(k+1))\ (2k+1)\ (k-2)\ (1+\beta(k+3))},$$

and where the polynomial P_m are defined as

$$\begin{aligned}
P_0(X) &= 2X^4 + 5X^3 + 16X^2 - 8,\\
P_1(X) &= 4X^5 + 18X^4 + 40X^3 + 68X^2 - 28X - 56,\\
P_2(X) &= 2X^6 + 21X^5 + 42X^4 + 35X^3 + 16X - 112,\\
P_3(X) &= 8X^6 + 18X^5 + 24X^4 - 68X^3 - 144X^2 - 112X - 64.
\end{aligned}$$

Local optimality

Corollary

If B is a ball in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $t \mapsto B_t = (I + t\mathbf{V} + O(t^2))(B)$ a smooth volume preserving deformation, then

$$\lambda_{1,\beta}(B) \geq \lambda_{1,\beta}(B_t), \quad \textit{for } t \textit{ small enough.}$$

Perspectives - Open Problems

- Positive answer to the question when $\Omega \subset \mathbb{R}^3$ and $\partial\Omega$ is differomorphic to $\mathbb{S}^2$?
- Study the stability question for Hersch's inequality in a smooth neighborhood :
  - solve the "two-norm discrepancy issue" :

$$S(\Omega) := \sum_{i=1}^{d} \frac{1}{\lambda_i^{LB}(\Omega)} = S(B) + \underbrace{S''(B)(V,V)}_{\leq -\alpha \|V_n\|_{H^1}^2} + o(\|V\|_{C^3}^2).$$

  - prove coercivity for deformation preserving the perimeter.
  - Enlarge the neighborhood/regularization procedure.