Cavity detection by boundary measurements

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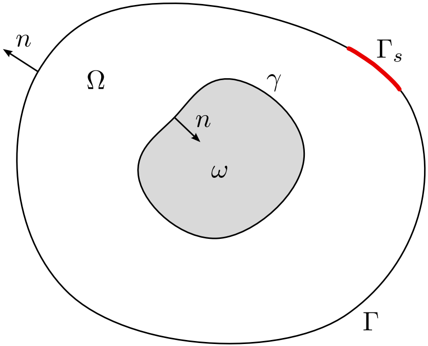
Statement of the problem



Potential function:

$$
\begin{aligned}
-\Delta\varphi^F &= 0 \quad \text{in } \Omega \setminus \omega \\
\partial_n \varphi^F &= F \quad \text{on } \Gamma \\
\partial_n \varphi^F &= 0 \quad \text{on } \gamma
\end{aligned}
$$

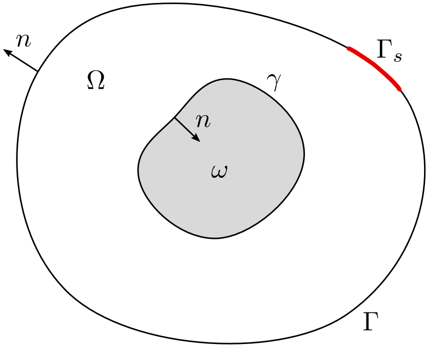
with F such that:

$$\int_\Gamma F \, \mathrm{d}s = 0.$$

Is it possible to reconstruct ω, knowing the NtD map

$$\Lambda^* : F \in H^{-1/2}(\Gamma) \mapsto \varphi^F|_\Gamma \in H^{1/2}(\Gamma) \ ?$$

Statement of the problem



Stream function:

$$
\begin{aligned}
-\Delta \psi^f &= 0 \quad \text{in } \Omega \setminus \omega \\
\psi^f &= f \quad \text{on } \Gamma \\
\psi^f &= c \quad \text{on } \gamma
\end{aligned}
$$

with $f = \partial_\tau F$ and $c \in \mathbb{R}$ such that:

$$\int_\gamma \partial_n \psi^f \, \mathrm{d}s = 0.$$

Is it possible to reconstruct ω, knowing the DtN map

$$\Lambda : f \in H^{1/2}(\Gamma) \mapsto \partial_n \psi^f|_\Gamma \in H^{-1/2}(\Gamma) \ ?$$

More generally

- Inverse Calderón problem (or inverse conductivity problem or electrical impedance tomography):

$$\begin{aligned} \mathrm{div}(\gamma \nabla u) = 0 \quad & \text{in } \Omega, \\ u \text{ (or } \partial_n u) = f \quad & \text{on } \partial\Omega. \end{aligned}$$

  Determine the fonction γ by means of the knowledge of

$$\Lambda : f \in H^{1/2}(\Gamma) \mapsto \partial_n u \in H^{-1/2}(\Gamma).$$

- The case:

$$\gamma = 1 + (k-1)\chi_\omega,$$

  with $0 < k < 1$ and χ_ω the characteristic function of ω corresponds to an inclusion ω.
- The degenerate case $k = 0$, corresponds to a cavity.
- Plenty of variations: boundary conditions on $\partial\omega$, cracks,...

Issues

Main issues:

- Identifiability (or well-posedness): is the relation $\Lambda \leftrightarrow \omega$? one-to-one?
- Stability: do “close” measurements correspond to “close” ω?
- Reconstruction: How to determine ω from the measurements?

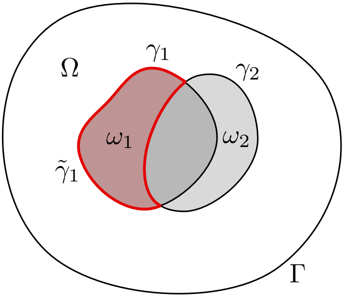
Auxillary issues:

- Same as in the first list but with only partial measurements (on Γ_s).
- How many measures are needed to identify and reconstruct ω?

Some references

- Calderón (1980)
- Bukhgeim, Cheng and Yamamoto (1999): stability for one single cavity.
- Alessandrini and Rondi (2001): optimal stability for the multi-cavities problem.
- Akduman, Haddar and Kress (2002, 2004, 2005): reconstruction a a single cavity with conformal mappings.
- Allessandrini, Morassi and Rosset (2002): stable determination of the diameter of a cavity.
- Dambrine and Kateb (2007): improvement of the work of Akduman, Haddar and Kress.
- Rundell (2008): reconstruction for one single starlike cavity (with Robin boundary condition).
- Caubet, Dambrine, Kateb (2013): reconstruction of a cavity based on shape optimization technics

Identifiability



Straightforward arguments requiring only one (non-constant) measurement:

- Assume that two cavities give the same measurement.
- Define $\psi = \psi_1 - \psi_2$.
- $\Delta\psi = 0$ and $\partial_n\psi = \psi = 0$ on Γ, hence $\psi = 0$.
- $c_1 = c_2 = c$ and $\psi_2 = c$ on $\tilde{\gamma}_1$. Then $\psi_2 = c$ in Ω.

Framework

- The sets Ω and ω are assumed to be $C^{1,1}$.
- Rather that Λf, we will consider the quantities:

$$\langle \Lambda f, g \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \quad (f \in H^{1/2}(\Gamma),\, g \in H^{1/2}(\Gamma)).$$

- Let ϕ be a conformal mapping from D (the unit disk) onto Ω. Then:

$$\langle \Lambda f^\dagger, g^\dagger \rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)} = \langle \Lambda f, g \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},$$

where $f^\dagger = f \circ \phi$ and $g^\dagger = f \circ \phi$.

$\rightarrow$ We can assume that Ω is a disk.

Single layer potential

Definitions

Let G be defined by:

$$G(x) = -\frac{1}{2\pi} \ln |x|, \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

The single layer potential associated with the density $q \in H^{-1/2}(\Gamma)$ is:

$$\mathcal{S}_\Gamma q(x) = \int_\Gamma G(x-y)q(y)\,\mathrm{d}\sigma_y, \qquad x \in \mathbb{R}^2 \setminus \Gamma.$$

We denote

$$u_i(x) = \mathcal{S}_\Gamma q(x) \quad x \in \Omega$$
$$u_e(x) = \mathcal{S}_\Gamma q(x) \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}.$$

The single layer potential $u = \mathcal{S}_\Gamma q$ is the unique solution to the transmission problem:

$$\begin{aligned} -\Delta u &= 0 && \text{in } \mathbb{R}^2 \setminus \Gamma \\ [u] &= 0 && \text{on } \Gamma \\ [\partial_n u] &= q && \text{on } \Gamma \\ u &= \mathcal{O}(\ln|x|) && \text{as } |x| \to +\infty, \end{aligned}$$

where

$$\begin{aligned} [u] &= u_i - u_e && \text{on } \Gamma \\ [\partial_n u] &= \partial_n u_i - \partial_n u_e && \text{on } \Gamma. \end{aligned}$$

Single layer potential

Isomorphism

Denote by $\mathsf{S}_\Gamma q$ the trace of $\mathcal{S}_\Gamma q$ on Γ. It can be shown that:

$$\mathsf{S}_\Gamma q(x) = \int_\Gamma G(x-y)q(y)\,\mathrm{d}\sigma_y, \qquad x \in \Gamma.$$

The linear operator:

$$\mathsf{S}_\Gamma : q \in H^{-1/2}(\Gamma) \mapsto \mathsf{S}_\Gamma q \in H^{1/2}(\Gamma),$$

is an isomorphism.
In particular, for every $f \in H^{1/2}(\Gamma)$, the Dirichlet problem:

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{on } \Gamma, \end{aligned}$$

is equivalent to the integral equation:

$$\mathsf{S}_\Gamma q = f \text{ on } \Gamma.$$

Restatement of the detection problem

In term of single layer potentials, we get:

$$\begin{aligned} \mathsf{S}_\gamma q_\gamma + \mathcal{S}_\Gamma q_\Gamma|_\gamma &= c \quad \text{on } \gamma \\ \mathcal{S}_\gamma q_\gamma|_\Gamma + \mathsf{S}_\Gamma q_\Gamma &= f \quad \text{on } \Gamma \end{aligned}$$

where $c \in \mathbb{R}$ is such that:

$$\int_\gamma q_\gamma \, \mathrm{d}\sigma = 0.$$

The measure (normal derivative on Γ) is:

$$\frac{1}{2} q_\Gamma(x) + \int_\Gamma \nabla G(x-y) \cdot n(x) q_\Gamma(y) \, \mathrm{d}\sigma_y + \int_\gamma \nabla G(x-y) \cdot n(x) q_\gamma(y) \, \mathrm{d}\sigma_y.$$

Asymptotic expansion

Let Ω and ω be given and $r \in \Omega$, $\varepsilon \geqslant 0$ and denote:

$$\omega_r^\varepsilon = r + \varepsilon(\omega - r).$$

We consider the DtN map corresponding to the cavity ω_r^ε:

$$\Lambda_r^\varepsilon : f \in H^{1/2}(\Gamma) \mapsto \partial_n \psi|_\Gamma \in H^{-1/2}(\Gamma).$$

Theorem

For every $r \in \Omega$, the DtN map Λ_r^ε can be expanded as a power series in ε in $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$:

$$\Lambda_r^\varepsilon = \Lambda^0 + \sum_{k \geqslant 2} \varepsilon^k \Lambda_r^k,$$

with non zero radius of convergence.

Asymptotic expansion

Expression of the first term

We have:

$$\langle \Lambda^0 f, g \rangle = \int_\Gamma \partial_n u^f \, g \, \mathrm{d}\sigma,$$

where

$$\begin{aligned} -\Delta u^f &= 0 \quad \text{in } \Omega \\ u^f &= f \quad \text{on } \Gamma. \end{aligned}$$

The radius of convergence of the other terms require more material...

Single layer potential

Equilibrium density

The equilibrium density $\psi_{\mathrm{eq}}^{\Gamma} \in H^{-1/2}(\Gamma)$ is the unique solution to:

$$\mathsf{S}_\Gamma \psi_{\mathrm{eq}}^{\Gamma} = c$$
$$\int_\Gamma \psi_{\mathrm{eq}}^{\Gamma} \, \mathrm{d}\sigma = 1.$$

The constant:

$$\mathrm{Cap}(\Gamma) = e^{2\pi c},$$

is the logarithmic capacity of Γ.

Single layer potential

Norms on $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$

- The following bilinear form is a scalar product in $H^{-1/2}(\Gamma)$:

$$\langle p, q \rangle_{H^{-1/2} \times H^{-1/2}} = \int_\Gamma \mathsf{S}_\Gamma p(y) q(y) \, \mathrm{d}\sigma_y .$$

  Notice that:

$$\langle p, q \rangle_{H^{-1/2} \times H^{-1/2}} = \int_{\mathbb{R}^2} \nabla \mathcal{S}_\Gamma q \cdot \nabla \mathcal{S}_\Gamma p \, \mathrm{d}x,$$

  providing that $\|\nabla \mathcal{S}_\Gamma p\|_{L^2(\mathbb{R}^2)} < +\infty$ and $\|\nabla \mathcal{S}_\Gamma q\|_{L^2(\mathbb{R}^2)} < +\infty$.
- The following bilinear form is a scalar product in $H^{1/2}(\Gamma)$:

$$\langle f, g \rangle_{H^{1/2} \times H^{1/2}} = \langle \mathsf{S}_\Gamma^{-1} f, S_\Gamma^{-1} g \rangle_{H^{-1/2} \times H^{-1/2}} .$$

Single layer potential

Poincaré's variational problem

It can be shown that:

$$\langle p, \psi_{\mathrm{eq}}^{\Gamma} \rangle_{H^{-1/2} \times H^{-1/2}} = 0 \quad \Leftrightarrow \quad \|\nabla \mathcal{S}_{\Gamma} p\|_{L^2(\mathbb{R}^2)} < +\infty.$$

Define $\Omega_e = \mathbb{R}^2 \setminus \overline{\Omega}$ and (Poincaré's variational problem):

$$\lambda_{\Gamma}^{+} = \max_{p \perp \psi_{\mathrm{eq}}^{\Gamma}} \frac{\|\nabla \mathcal{S}_{\Gamma} p\|_{L^2(\Omega_e)}^2 - \|\nabla \mathcal{S}_{\Gamma} p\|_{L^2(\Omega)}^2}{\|\nabla \mathcal{S}_{\Gamma} p\|_{L^2(\mathbb{R}^2)}^2}$$

$$\lambda_{\Gamma}^{-} = \min_{p \perp \psi_{\mathrm{eq}}^{\Gamma}} \frac{\|\nabla \mathcal{S}_{\Gamma} p\|_{L^2(\Omega_e)}^2 - \|\nabla \mathcal{S}_{\Gamma} p\|_{L^2(\Omega)}^2}{\|\nabla \mathcal{S}_{\Gamma} p\|_{L^2(\mathbb{R}^2)}^2}$$

Then the max and min are attained for orthogonal densities and $\lambda_{\Gamma}^{-} = -\lambda_{\Gamma}^{+}$ [Khavinson, Putinar & Shapiro, 2007].

Radius of convergence

Considering the asymptotic expansion of the DtN map Λ_r^ε, we have:

Theorem

The radius of convergence is a decreasing function of λ_Γ^+, λ_γ^+ *and* $d(r,\gamma)/d(r,\Gamma)$. *In particular, for a cavity small enough, the radius is larger than* 1.

Generalized Pólia-Szegö matrices

Define the harmonic polynomials of degree $n \geqslant 1$:

$$P_1^n(z) = \Re(z^n) + c_1^n$$
$$P_2^n(z) = \Im(z^n) + c_2^n,$$

where the constants are such that:

$$\langle P_1^n, \psi_{\text{eq}}^\gamma \rangle_{H^{1/2}(\gamma)\times H^{-1/2}(\gamma)} = \langle P_2^n, \psi_{\text{eq}}^\gamma \rangle_{H^{1/2}(\gamma)\times H^{-1/2}(\gamma)} = 0.$$

For every integers ℓ, ℓ', the Pólia-Szegö matrices of order $\ell + \ell'$ are:

$$\mathbb{M}_\gamma^{\ell,\ell'} = \left(\langle P_j^\ell, P_k^{\ell'} \rangle_{H^{1/2}(\gamma)\times H^{1/2}(\gamma)} \right)_{1 \leqslant j,k \leqslant 2}.$$

First introduced in [Pólia and Szegö, 1951] (order 1). Generalized polarization tensors: [Ammari et al. 2013].

Generalized Kirchhoff-Routh matrices

- Let $\psi_\Omega(\cdot, y)$ $(y \in \Omega)$ be the solution of:

$$\begin{aligned} -\Delta\psi_\Omega(\cdot, y) &= 0 && \text{in } \Omega \\ \psi_\Omega(\cdot, y) &= G(\cdot - y) && \text{on } \Gamma. \end{aligned}$$

  The function ψ_Ω is symmetric in x and y.
- For any harmonic function φ in Ω and for every integer k, we define:

$$\nabla^{(k)}\varphi = (\Re(f^{(k)}), -\Im(f^{(k)})),$$

  where f is holomorphic in Ω and such that $\Re(f) = \varphi$.
- For every integers ℓ, ℓ' and every $r \in \Omega$, the Kirchhoff-Routh matrices of order $\ell + \ell'$ are:

$$\mathbb{N}_r^{\ell,\ell'} = \nabla_x^{(\ell)} \otimes \nabla_y^{(\ell')} \psi_\Omega(r, r).$$

The detection problem

Expression of the other terms in the expansion

Denote, for every $k \geqslant 1$:

$A_k = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \backslash \{0\})^n,\ n \in \mathbb{N} \backslash \{0\},\ n \text{ even}, \alpha_1 + \ldots + \alpha_n = k\}.$

Then, for every $k \geqslant 2$, we have:

$$\langle \Lambda_r^k f, g \rangle =$$
$$\sum_{\alpha \in A_k} \frac{1}{\alpha!} \nabla^{(\alpha_1)} u^f(r) \mathbb{M}_\gamma^{\alpha_1, \alpha_2} \mathbb{N}_r^{\alpha_2, \alpha_3} \mathbb{M}_\gamma^{\alpha_3, \alpha_4} \ldots \mathbb{N}_r^{\alpha_{n-2}, \alpha_{n-1}} \mathbb{M}_\gamma^{\alpha_{n-1}, \alpha_n} \nabla^{(\alpha_n)} u^g(r).$$

Examples

The expressions of the firsts terms are:

$$\Lambda_r^2(f,g) = \nabla u^f(r)\mathbb{M}_\gamma^{1,1}\nabla u^g(r)$$

$$\Lambda_r^3(f,g) = \frac{1}{2!}\nabla^{(2)}u^f(r)\mathbb{M}_\gamma^{2,1}\nabla u^g(r) + \frac{1}{2!}\nabla u^f(r)\mathbb{M}_\gamma^{1,2}\nabla^{(2)}u^g(r)$$

$$\begin{aligned}\Lambda_r^4(f,g) &= \frac{1}{3!}\nabla^{(3)}u^f(r)\mathbb{M}_\gamma^{3,1}\nabla u^g(r) + \frac{1}{2!2!}\nabla^{(2)}u^f(r)\mathbb{M}_\gamma^{2,2}\nabla^{(2)}u^g(r)\\ &+ \frac{1}{3!}\nabla u^f(r)\mathbb{M}_\gamma^{1,3}\nabla^{(3)}u^g(r) + \nabla u^f(r)\mathbb{M}_\gamma^{1,1}\mathbb{N}_r^{1,1}\mathbb{M}_\gamma^{1,1}\nabla u^g(r).\end{aligned}$$

Comments

- The (unknown) geometry of γ is encoded in the Pólia-Szegö matrices.
- The (known) geometry of Γ is encoded in the Kirchhoff-Routh matrices.
- The alternance of the Pólia-Szegö matrices and the Kirchhoff-Routh matrices can be thought of as the "bounce back" of the geometrical information on the boundaries Γ and γ.

Toward a reconstruction method

- How to access the PS matrices from the measurements?
- How to access the geometry of ω from the PS matrices?

From now on, ε is no longer assumed to be small.

From measurements to PS matrices

The measurements

We can recombine the terms as follows:

$$\langle \Lambda_r^\varepsilon f, g\rangle - \langle \Lambda^0 f, g\rangle = \sum_{\ell,\ell'\geqslant 1} \frac{\varepsilon^{\ell+\ell'}}{\ell!\ell'!} \nabla^{(\ell)} u^f(r) \Sigma_\varepsilon^{\ell,\ell'} \nabla^{(\ell')} u^g(r),$$

where the measurement matrices reads:

$$\boxed{\Sigma_\varepsilon^{\ell,\ell'} = \mathbb{M}_\gamma^{\ell,\ell'} + \sum_{\substack{k\geqslant 1\\ \alpha\in A_k}} \frac{\varepsilon^k}{\alpha!} \mathbb{M}_\gamma^{\ell,\alpha_1} \mathbb{N}_r^{\alpha_1,\alpha_2} \mathbb{M}_\gamma^{\alpha_2,\alpha_3} \dots \mathbb{N}_r^{\alpha_{n-1},\alpha_n} \mathbb{M}_\gamma^{\alpha_n,\ell'}}$$

We have:

$$\nabla^{(k)} P_1^k = k!(1,0), \qquad \nabla^{(k)} P_2^k = k!(0,1),$$

and for $n \neq k$ and $j = 1, 2$:

$$\nabla^{(n)} P_j^k(0) = (0,0),$$

so we can access every matrix $\Sigma_\varepsilon^{\ell,\ell'}$.

From measurements to PS matrices

Getting rid of the KR matrices

We assume that Γ is a circle of radius $R > 1$. Then:

$$\psi_\Omega(x,y) = -\frac{1}{2\pi}\ln|R - x\bar{y}/R| = -\Re\left(\frac{1}{2\pi}\ln(R - x\bar{y}/R)\right).$$

We choose $r = 0$. We get that, for every $\ell, \ell' \geq 1$

- If $\ell \neq \ell'$ then:
$$\mathbb{N}_0^{\ell,\ell'} = 0,$$
- if $\ell = \ell'$:
$$\mathbb{N}_0^{\ell,\ell} = \frac{(\ell-1)!\ell!}{2\pi}\frac{1}{R^{2\ell}}\mathrm{Id}.$$

In that case, the measurement matrices simplify as:

$$\Sigma_\varepsilon^{\ell,\ell'} = \mathbb{M}^{\ell,\ell'} + \sum_{\substack{n\geq 1\\ \alpha\in(\mathbb{N}\setminus\{0\})^n}} \frac{\varepsilon^{|\alpha|}}{(2\pi)^n R^{2|\alpha|}\alpha_1\ldots\alpha_n}\mathbb{M}_\gamma^{\ell,\alpha_1}\mathbb{M}_\gamma^{\alpha_1,\alpha_2}\ldots\mathbb{M}_\gamma^{\alpha_n,\ell'}.$$

In particular, when $\ell' = 1$, we get:

$$\Sigma_\varepsilon^{\ell,1} = \mathbb{M}_\gamma^{\ell,1} + \sum_{k\geq 1} \frac{\varepsilon^k}{2\pi k R^{2k}} \Sigma_\varepsilon^{\ell,k} \mathbb{M}_\gamma^{k,1}.$$

Let $\ell^2(\mathcal{M}_2)$ be the space of the sequences $(\mathbb{K}_k)_{k\geq 1}$ of 2×2 matrices such that:

$$\sum_{k\geq 1} \frac{\varepsilon^k}{2\pi k R^{2k}} \mathrm{Tr}(\mathbb{K}_k \mathbb{K}_k^t) < +\infty.$$

We have:

$$(\Sigma_\varepsilon^{\ell,1})_{\ell\geqslant 1} = (\mathrm{Id} + \Pi_\varepsilon)((\mathbb{M}_\gamma^{\ell,1})_{\ell\geqslant 1}),$$

with

$$\Pi_\varepsilon((\mathbb{M}_\gamma^{\ell,1})_{\ell\geqslant 1}) = \sum_{k\geq 1} \frac{\varepsilon^k}{2\pi k R^{2k}} \Sigma_\varepsilon^{\ell,k} \mathbb{M}_\gamma^{k,1}.$$

From measurements to PS matrices

Final step

Lemma

Providing that λ_γ^+ *and* $d(0,\gamma)/d(0,\Gamma)$ *are small enough, the linear operator* Π_ε *from* $\ell^2(\mathcal{M}_2)$ *into itself has a norm lower than 1.*

- Under lemma's smallness assumptions, the linear operator $\mathrm{Id} + \Pi_\varepsilon$ is invertible.
- The PS matrices do no depend linearly (but quadratically) on the measurement matrices (Π_ε depend on the measurement matrices).
- We can get a stability result at this stage.
- We get a natural numerical scheme by simply replacing the sequence of measurement matrices by a truncated version.

From the PS matrices to the reconstruction

The Pólia-Szegö matrices can be computed by considering the conformal mapping:

$$\phi(z) = a_1 z + a_0 + \sum_{\ell \leqslant 1} a_\ell z^\ell,$$

that maps the exterior of the unit disk onto the exterior of ω.
For $n \geqslant 1$ and $\ell \leqslant n$, define $a_\ell^n \in \mathbb{C}$ such that:

$$\phi(z)^n = \sum_{\ell \leqslant n} a_\ell^n z^\ell.$$

Then, we have:

$$\mathbb{M}_\gamma^{\ell,\ell'} = 2\pi \begin{pmatrix} \Re\left(\sum_{\ell\geq 1} \ell(\bar{a}_\ell^n a_\ell^k + a_\ell^n a_{-\ell}^k)\right) & -\Im\left(\sum_{\ell\geq 1} \ell(\bar{a}_\ell^n a_\ell^k - a_\ell^n a_{-\ell}^k)\right) \\ \Im\left(\sum_{\ell\geq 1} \ell(\bar{a}_\ell^n a_\ell^k + a_\ell^n a_{-\ell}^k)\right) & \Re\left(\sum_{\ell\geq 1} \ell(\bar{a}_\ell^n a_\ell^k - a_\ell^n a_{-\ell}^k)\right) \end{pmatrix}.$$

From the PS matrices to the reconstruction

For instance, using symbolic computation (Sage):

$$\mathbb{M}_\gamma^{1,1} = 2\pi a_1 \begin{pmatrix} a_1 + \Re(a_{-1}) & \Im(a_{-1}) \\ \Im(a_{-1}) & a_1 - \Re(a_{-1}) \end{pmatrix},$$

and

$$\mathbb{M}_\gamma^{2,1} = 2\pi a_1 \begin{pmatrix} \Re(a_0 a_{-1} + a_1 a_{-2} + a_1 a_0) & \Im(a_0 a_{-1} + a_1 a_{-2} - a_1 a_0) \\ \Im(a_0 a_{-1} + a_1 a_{-2} + a_1 a_0) & -\Re(a_0 a_{-1} - a_1 a_{-2} + a_1 a_0) \end{pmatrix}.$$

More generally, we have:

$$\mathbb{M}_\gamma^{k,n} = 2\pi a_1^{n+k-1} \begin{pmatrix} \Re(a_{1-k-n}) & \Im(a_{1-k-n}) \\ \Im(a_{1-k-n}) & -\Re(a_{1-k-n}) \end{pmatrix} + \tilde{\mathbb{M}}_\gamma^{k,n},$$

where $\tilde{\mathbb{M}}_\gamma^{k,n}$ depends only on $a_1, a_0, \dots, a_{2-k-n}$.

Geometric decoupling as $\varepsilon \to 0$

Numerical simulations

Algorithm

1. Use the harmonic polynomials to access the measurements matrices $\Sigma_\varepsilon^{\ell,\ell'}$.
2. Inverse the linear $2m \times 2m$ system:

$$\begin{pmatrix} \Sigma_\varepsilon^{1,1} \\ \Sigma_\varepsilon^{2,1} \\ \vdots \\ \Sigma_\varepsilon^{m,1} \end{pmatrix} = \begin{pmatrix} \mathbb{M}_\gamma^{1,1} \\ \mathbb{M}_\gamma^{2,1} \\ \vdots \\ \mathbb{M}_\gamma^{m,1} \end{pmatrix} + \frac{1}{2\pi} \begin{pmatrix} \frac{\varepsilon}{R^2}\Sigma_\varepsilon^{1,1} & \cdots & \frac{\varepsilon^m}{R^{2m}}\Sigma_\varepsilon^{1,m} \\ \frac{\varepsilon}{R^2}\Sigma_\varepsilon^{2,1} & \cdots & \frac{\varepsilon^m}{R^{2m}}\Sigma_\varepsilon^{2,m} \\ \vdots & \ddots & \vdots \\ \frac{\varepsilon}{R^2}\Sigma_\varepsilon^{m,1} & \cdots & \frac{\varepsilon^m}{R^{2m}}\Sigma_\varepsilon^{m,m} \end{pmatrix} \begin{pmatrix} \mathbb{M}_\gamma^{1,1} \\ \mathbb{M}_\gamma^{2,1} \\ \vdots \\ \mathbb{M}_\gamma^{m,1} \end{pmatrix}$$

   to get the PS matrices.
3. Extract the geometric information (i.e. a_1, a_0,$\ldots a_{-m}$) from the PS matrices.
4. Rebuild the cavity. The boundary is parameterized by:

$$s \in [-\pi, \pi[\mapsto \sum_{-m \leqslant k \leqslant 1} a_k e^{iks}.$$

Numerical simulations

To get a_1, a_0,... a_{-m} (i.e. $m + 2$ complex coefficients), $2m$ (real) measures are required.

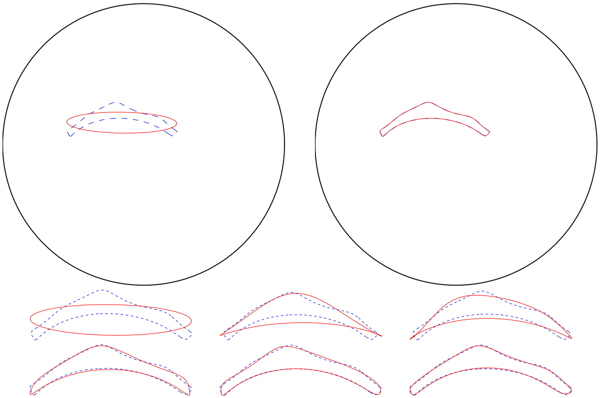


Figure : Localization and reconstruction of the cavity: 8 (complex) coefficients in the Riemann map are computed by means of 12 (real) measures.

Conclusion and To-do list

- Localized measurements: ok.
- More numerics.
- Stability: from the PS matrices to the geometry.
- More about λ_γ^+ (radius of convergence, invertibility of Π_ε).
- ...