Mean curvature bounds and eigenvalues of Robin Laplacians

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An eigenvalue problem

Context:

- $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain.
- Eigenvalue problem: Find (λ, u) such that

$$\begin{cases} -\Delta u = \lambda u & \text{on} \quad \Omega, \\ \partial_n u - \alpha u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- $\alpha \in \mathbb{R}$ and ∂_n the outward derivative.

Associated operator:

$$q_\alpha^\Omega(u) = \int_\Omega |\nabla u|^2 - \alpha \int_{\partial\Omega} |u|^2 \mathrm{d}S \equiv q_\alpha(u), \quad u \in H^1(\Omega).$$

- $H_\alpha^\Omega \equiv H_\alpha$ the associated self-adjoint operator.
- $E_j(\alpha, \Omega) \equiv E_j(\alpha)$ the j-th eigenvalue of H_α^Ω.

Faber-Krahn inequalities

Question: Among domains of fixed volume, how to optimize the first eigenvalues of the Laplacian?

- For the Dirichlet Laplacian, the balls are the minimizers of the first one.
- For the Robin Laplacian with $\alpha < 0$, the balls are the minimizers among Lipschitz domains ([Bossel 86] for the 2d case, [Daners 06] in general).

Basic properties for $\alpha \geq 0$ and Lipschitz domains:

- If $\alpha > 0$, then $E_1(\alpha) < 0$.
- $\alpha \mapsto E_j(\alpha)$ is concave, decreasing on $\mathbb{R}$.
- $\frac{\partial E_1}{\partial \alpha}(0) = -\frac{|\partial\Omega|}{|\Omega|}$.
- $E_j(\alpha) \to -\infty$ when $\alpha \to +\infty$ (to be continued).

Conjecture ([Barecket 77]): For $\alpha > 0$, the balls are the maximizers for $E_1(\alpha)$.

The counter-example of annuli

Proposition ([Freitas-Krejkerik 14])

Let $\mathcal{A}_{r_1,r_2} \subset \mathbb{R}^n$ be the annulus of radii $0 \leq r_1 < r_2$ (a ball for $r_1 = 0$). Then

$$E_1(\alpha, \mathcal{A}_{r_1,r_2}) \underset{\alpha\to+\infty}{=} -\alpha^2 - \frac{n-1}{r_2}\alpha + o(\alpha).$$

Element of proof:

- Look at the axisymmetrical problem

$$\begin{cases} -r^{n-1}(r^{n-1}u')' = Eu, & r \in (r_1, r_2), \\ -u'(r_1) + \alpha u(r_1) = 0 \text{ and } u'(r_2) + \alpha u(r_2) = 0. \end{cases}$$

- Use Bessel's functions and their expansions at ∞.

Corollary

Let $R > 0$ and $\mathcal{B}_R$ the associate centered ball of radius R. Let $\mathcal{A}_{r_1,r_2}$ be the annulus of the same volume of $\mathcal{B}_R$. Then for α large enough

$$E_1(\alpha, \mathcal{B}_R) < E_1(\alpha, \mathcal{A}_{r_1,r_2}).$$

Comparison with an annulus for general domain

Auxiliary annulus:

- Let $\Omega \subset \mathbb{R}^2$, and define

$$r_1 = \frac{\sqrt{|\partial\Omega|^2 - 4\pi|\Omega|}}{2\pi} \quad \text{and} \quad r_2 = \frac{|\partial\Omega|}{2\pi}.$$

- Notice that $|\Omega| = |\mathcal{A}_{r_1,r_2}|$.
- First eigenvalue of the Laplacian with artificial Neumann b.c. at $r = r_1$:

$$\mu_1(\alpha, \mathcal{A}_{r_1,r_2}) = \inf_{\substack{\psi \in H^1(r_1,r_2) \\ \psi \neq 0}} \frac{\int_{r_1}^{r_2} \psi'(r)^2 r\,\mathrm{d}r + \alpha r_2 \psi(r_2)^2}{\|\psi\|_2^2}.$$

Following a change of variables from [Payne-Weinberger 61]:

Lemma ([Freitas-Krejčeřík 14])

Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^2$ domain. Then

$$\forall \alpha \geq 0, \quad E_1(\alpha, \Omega) \leq \mu_1(\alpha, \mathcal{A}_{r_1,r_2}).$$

Small parameter in dimension 2

Theorem ([Freitas-Krejčeřík 14])

Let $\omega > 0$. There exists $\alpha^* > 0$ such that for all domain $\Omega \subset \mathbb{R}^2$ of class $\mathcal{C}^2$ with $|\Omega| = \omega$, there holds

$$\forall \alpha \in [0, \alpha_*], \quad E_1(\alpha, \Omega) \leq E_1(\alpha, \mathcal{B}_R), \tag{1}$$

where $\mathcal{B}_R$ is a ball of volume ω.

Ideas of the proof

- Consider $R > 0$ fixed so that $r_1 = \sqrt{2\epsilon R + R^2}$ and $r_2 = R + \epsilon$ with $\epsilon > 0$.
- When $\alpha \to 0$, notice the expansions

$$\begin{cases} E_1(\alpha, \mathcal{B}_{r_3}) = -2\alpha \dfrac{R}{R^2} + O(\alpha^2), \\ \mu_1(\alpha, \mathcal{A}_{r_1,r_2}) = -2\alpha \dfrac{r_2}{R^2} + O(\alpha^2), \end{cases} \quad \text{and use } r_2 > R.$$

- Deduce $\alpha(\epsilon) > 0$ such that (1) holds on $[0, \alpha(\epsilon)]$.
- Check that $\epsilon \mapsto \alpha(\epsilon)$ is bounded from below by uniformity arguments.

Large parameter and first order asymptotics

Problematics

- Behavior of $E_j(\alpha)$ when $\alpha \to +\infty$.
- Influence on the geometry of Ω:
  Singularities of the boundary, curvature, symmetries.
  Localization of the eigenfunctions as $\alpha \to +\infty$.

In general

- Ω a corner domain (Levitin and Parnovski 08):

$$E_1(\alpha) \underset{\alpha \to +\infty}{=} C(\Omega)\alpha^2 + o(\alpha^2)$$

- Ω regular: $C(\Omega) = -1$ ([Lacey-Ockendo-Sabina 99], [Daners-Kennedy 08], [Lu-Zhu 04]). Asymptotics valid for all $j \geq 1$.

2D polygonal domains:

- Ω is a polygonal with opening angles $\alpha_k \in (0, \pi)$:

$$C(\Omega) = -\max_k \sin^{-2}\left(\frac{\alpha_k}{2}\right)$$

- Tunneling in case of symmetries ([Helffer-Pankrashkin 14]).
- Exterior of a convex polygon ([Pankrashkin 14]).

Regular case

State of the art for the regular case

- Ω is 2D, regular, and κ_{max} the maximum of the curvature at the boundary. [Exner-Minakov-Parnovski 13], [Pankrashkin 13] (included non convex corners):

$$E_j(\alpha) \underset{\alpha\to+\infty}{=} -\alpha^2 - \kappa_{\text{max}}\alpha + O(\alpha^{2/3})$$

- Spherical Shells and balls in any dimension ([Freitas-Krejčeřík 14]).
- In 2D: Precise asymptotics under geometrical assumptions on the curvature:

Theorem ([Helffer-Kachmar 14])

Assume that $\Omega \subset \mathbb{R}^2$ is $\mathcal{C}^\infty$ and that the curvature κ has a unique non degenerate maximum at $s_0 \in \partial\Omega$, then for all $j \geq 1$, for all $N \geq 0$,

$$E_j(\alpha) \underset{\alpha\to+\infty}{=} -\alpha^2 - \alpha\kappa_{\text{max}} + (2j-1)\sqrt{\frac{|\kappa''(s_0)|}{2}}\alpha^{1/2} + \sum_{k=0}^{N} \gamma_{j,k}\alpha^{-\frac{j}{2}} + o(\alpha^{-\frac{N}{2}}),$$

Questions:

What is the result in the n-dimensional case? What plays the role of κ_{max}?

Main result

Theorem [Pankrashkin-P. 15]

Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^2$ domain with compact boundary, and H the mean curvature of the boundary. Let $-\Delta^S$ be the Laplace-Beltrami operator on $\partial\Omega$ and $\lambda_j(\alpha)$ the j-th eigenvalue of the operator $-\Delta^S - \alpha(n-1)H$. Then for all $j \geq 1$:

$$E_j(\alpha) \underset{\alpha \to +\infty}{=} -\alpha^2 + \lambda_j(\alpha) + O(\log \alpha).$$

Technics:

- Dirichlet To Neumann bracketing.
- Tubular coordinates and estimates of the metrics.
- Localization near the boundary and study of 1D operators.

Corollary

Assume that Ω is $\mathcal{C}^2$. Let $H_{\max}$ be the maximum of the mean curvature. Then

$$\forall j \geq 1, \quad E_j(\alpha) \underset{\alpha \to +\infty}{=} -\alpha^2 - (n-1)H_{\max}\alpha + o(\alpha).$$

If the boundary is $\mathcal{C}^3$ (resp., $\mathcal{C}^4$), then the remainder can be replaced by $O(\alpha^{2/3})$ (resp., $O(\alpha^{1/2})$).

Change of variables

Tubular coordinates:

- $\Sigma = S \times (0, \delta)$. For δ small, we have a diffeomorphism:

$$\Phi : \Sigma \mapsto \Omega_\delta \text{ defined by } \Phi(s,t) = s - tn(s).$$

- $G = g \circ (\mathrm{Id} - tL_s)^2 + \mathrm{d}t^2$ the associate metric with

$$\begin{cases} g & \text{the embedded metric on } S \\ L_s = \mathrm{d}n(s) & \text{the shape operator} \end{cases}$$

New quadratic forms:

- Define $(G^{i,j}) = G^{-1}$ and $\varphi(s,t) = \det(\mathrm{Id} - tL_s)$.
- New quadratic forms acting on $L^2_G(\Omega_\delta) = \{u, \int_\Sigma |u|^2 \varphi \mathrm{d}\Sigma < +\infty\}$:

$$k^{\star,\delta}_\alpha(u) := \int_\Sigma \Big(\sum_{i,j} G^{i,j} \partial_i u \, \partial_j u \Big) \varphi(s,t) \mathrm{d}\Sigma - \alpha \int_S |u(s,0)|^2 \mathrm{d}S, \quad \star = \mathrm{D}, \mathrm{N}$$

- Natural (weighted) domains follow through the change of variables.
- Unitary transform: $E_j(B^{\star,\delta}_\alpha) = E_j(K^{\star,\delta}_\alpha)$.

Application to Faber-Krahn inequality

Corollary

Let Ω_1 and Ω_2 be two regular connected domains such that $H_{\max}(\Omega_1) < H_{\max}(\Omega_2)$. For all $j \geq 1$ there exists $\alpha_0 > 0$ such that

$$\forall \alpha \geq \alpha_0, \quad E_j(\alpha, \Omega_2) < E_j(\alpha, \Omega_1)$$

- Maximization of $E_j(\alpha)$ leads to the question:

How to minimize $H_{\max}(\Omega)$?

No minimizer in general (think of an annulus).

Proposition

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain different from a ball. Then for all $\epsilon > 0$ there exists Ω_ϵ smooth such that: $\mathrm{Vol}(\Omega) = \mathrm{Vol}(\Omega_\epsilon)$, $\partial\Omega$ is diffeomorphic to $\partial\Omega_\epsilon$, the Hausdorff distance between Ω_ϵ and Ω is smaller than ϵ, and

$$H_{\max}(\Omega_\epsilon) < H_{\max}(\Omega).$$

Minimization of the mean curvature

Theorem (Pankrashkin-P.)

Let $\Omega \subset \mathbb{R}^n$ be a bounded, star-shaped, $\mathcal{C}^2$ domain. Then

$$H_{\max}(\Omega) \geq \left(\frac{\mathrm{Vol}\,\mathcal{B}_n}{\mathrm{Vol}\,\Omega}\right)^{1/n}$$

with $\mathcal{B}_n$ the unit ball of $\mathbb{R}^n$. Moreover this is an equality if and only if Ω is a ball.

Corollary

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular star-shaped domain different from a ball. Let B be a ball of the same volume. Then for all $j \geq 1$ there exists α_0 such that

$$\forall \alpha \geq \alpha_0, \quad E_j(\alpha, \Omega) < E_j(\alpha, B).$$

- Open question: Does the theorem hold among domains with connected boundary?

Splitting: Analysis of the reduced operator

Semi-classical point of view

- Toward a Schrödinger operator:

$$-\Delta^S - \alpha(n-1)H = h^{-2}(-h^2\Delta^S + V) \text{ with } \begin{cases} h = \alpha^{-1/2}, \\ V = -(n-1)H \end{cases}$$

- Look at the minima of V (the maxima of H). If V is regular:

$$\lambda_j(\alpha) \underset{\alpha\to+\infty}{=} -(n-1)H_{\max}\alpha + o(\alpha).$$

Corollary (Using [Helffer-Sjöstrand, Simon, 80'])

Assume that Ω is $\mathcal{C}^5$, that H has a unique maximum at s_0, and that the Hessian at s_0 is non-degenerate. Denote by μ_k its eigenvalues. Then, when $\alpha \to +\infty$:

$$\lambda_j(\alpha) \underset{\alpha\to+\infty}{=} -(n-1)H_{\max}\alpha + e_j\alpha^{1/2} + O(\alpha^{3/4}).$$

where e_j is the j-th element of the set $\{\sum_1^{n-1} \sqrt{\frac{|\mu_k|}{2}}(2n_k - 1), (n_k)_k \in \mathbb{N}^{n-1}\}$.

To go further: degenerate cases

Corollary (Using [Martinez-Rouleux 88])

Assume that $\Omega \subset \mathbb{R}^2$ is $\mathcal{C}^{2p+3}$, and that the curvature has a unique maxima at s_0 of order $2p$ with $p \geq 1$:

$$H(s) = -C(s-s_0)^{2p} + O((s-s_0)^{2p+1}), \quad C > 0$$

Then, when $\alpha \to +\infty$:

$$\lambda_j(\alpha) \underset{\alpha\to+\infty}{=} -(n-1)H_{\max}\alpha + e_j\alpha^{\frac{1}{p+1}} + O(1)$$

where e_j is the j-th eigenvalue of $-\partial_s^2 + Cs^{2p}$ where $s \in \mathbb{R}$.

- Case of several maxima: each well creates its eigenvalues.
- If H is maximal on a set with non empty interior, then

$$\lambda_j(\alpha) \underset{\alpha\to+\infty}{=} -(n-1)H_{\max}\alpha + O(1).$$

perspectives

On the asymptotics for large α:

- Tunneling effect in case of symmetries (see the WKB construction from Helffer-Kachmar).
- Estimates on the eigenfunctions.
- Asymptotics when the domain has corners (see the method from Bonnaillie-Noël-Dauge-P. for the magnetic Laplacian).

On the Faber-Krahn inequality

- Prove the inequality for star-shaped domains for all $\alpha > 0$.
- Prove that for α large enough, the maximizer is an annulus (conjecture based on numerical simulations from [Antunes-Freitas-Kennedy 15]). The radii of the optimal annulus should depend on α.